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Superalgebras and the supersymmetric harmonic oscillator

M J Englefield

Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia

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Abstract. The bound states of the D-dimensional supersymmetric oscillator with a bosonic and fermionic component are shown to span an irreducible representation of osp$(3, 2D)$, which contains the supersymmetric Hamiltonian. The representation branches to two infinite-dimensional irreducible representations of osp$(2, 2D)$ and osp$(1, 2D)$ subalgebras. The one-dimensional case is used to exhibit S(2) subalgebras with representations, both finite and infinite dimensional, which are reducible but indecomposable. The supersymmetry associated with the radial equation of the D-dimensional oscillator is considered and states having a fixed angular momentum in each sector provide an irreducible representation space for an osp(2, 2) superalgebra and osp(1, 2) subalgebra.

1. Introduction

Although applications and illustrations of Lie algebras in elementary non-relativistic quantum mechanics are well known, the extension to superalgebras is limited. The initial work of de Crombrugghe and Rittenberg (1983) pointed out the association of osp$(1, 2D)$ with the harmonic oscillator, and indicated that supersymmetric systems could be expected to furnish further examples, since they always involve at least a three-dimensional superalgebra S(2). The D-dimensional supersymmetric harmonic oscillator was investigated, and a relevant osp$(2D, 2D)$ superalgebra given. The present paper extends this investigation to show that the supersymmetric oscillator provides representations of other superalgebras, and specific examples of representations of S(2) which are reducible but indecomposable. Most of the representations obtained are infinite dimensional. The supersymmetric systems are assumed to have just two components, bosonic and fermionic.

A subject of continuing interest is the relation between harmonic oscillators and systems with Coulomb potentials. It has been used by d'Hoker and Vinet (1985) to associate a superalgebra with a Coulomb potential, using an osp(2, 2) algebra associated with the four-dimensional oscillator.

Section 2 of this paper summarises the connection of the superalgebra S(2) with a one-dimensional supersymmetric system. In the next section an osp(3, 2) superalgebra is associated with the one-dimensional supersymmetric oscillator, by reinterpreting the work of van der Jeugt (1984) on the spin system. Subalgebras of this osp(3, 2) are then identified, and their representations on the oscillator states considered. The representations are reducible but indecomposable when the subalgebra is not semi-simple. In § 4 the one-dimensional results are generalised to D dimensions. In terms of algebras this amounts to replacing sp(2) by sp$(2D)$, so that $(r = 1, 2, 3)$ osp$(r, 2)$ becomes osp$(r, 2D)$. The two-dimensional system is considered in detail. By taking a new basis for the superalgebras that is related to polar coordinates rather than cartesian, an S(2) algebra is identified which relates to the one-dimensional supersymmetry on the radial equation that was given by Kostelecky *et al* (1985). Comparing this with the work on the one-dimensional oscillator in § 2 leads to an associated osp(2, 2) superalgebra of operators that act on the radial functions only.

After this work was completed the author saw a similar discussion of the one-dimensional case by Beckers *et al* (1987).

2. Supersymmetry and superalgebras

The basic connection between superalgebras and supersymmetry in elementary quantum mechanics was given by de Crombrugghe and Rittenberg (1983). If $A^{\pm} = 2^{-1/2}(\pm \mathrm{d}/\mathrm{d}x + U(x))$ then

$$Q' = \begin{bmatrix} 0 & A^+ \\ A^- & 0 \end{bmatrix} \qquad Q'' = \mathrm{i}\begin{bmatrix} 0 & -A^+ \\ A^- & 0 \end{bmatrix} \tag{2.1}$$

satisfies

$$\{Q', Q''\} = 0 = [Q', H_s] = [Q'', H_s] \tag{2.2}$$

where

$$\begin{aligned} Q'^2 = Q''^2 = H_s &= \frac{1}{2}\begin{bmatrix} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + U^2 + \dfrac{\mathrm{d}U}{\mathrm{d}x} & 0 \\ 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + U^2 - \dfrac{\mathrm{d}U}{\mathrm{d}x} \end{bmatrix} \\ &= \begin{bmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{bmatrix} \end{aligned} \tag{2.3}$$

is the supersymmetric Hamiltonian. The superalgebra S(2) is generated by the even operator H_s and the odd operators Q' and Q''. If $U(x)$ is an odd function, the eigenfunctions of H_s can be chosen to have definite parity, which determines the grading of the representation space.

Sukumar (1985) used the 'charge operators'

$$Q = \tfrac{1}{2}(Q' - \mathrm{i}Q'') = \begin{bmatrix} 0 & 0 \\ A^- & 0 \end{bmatrix} \qquad Q^{\dagger} = \tfrac{1}{2}(Q' + \mathrm{i}Q'') = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} \tag{2.4}$$

so that

$$\{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0 \qquad \{Q, Q^{\dagger}\} = H_s \tag{2.5}$$

and put

$$H_s = \begin{bmatrix} H - \varepsilon & 0 \\ 0 & \tilde{H} - \varepsilon \end{bmatrix} \tag{2.6}$$

where the 'factorisation energy' ε is related to U by

$$H\psi = \varepsilon\psi \qquad U = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi = \frac{1}{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x}. \tag{2.7}$$

The states

$$\begin{bmatrix} f(x) \\ 0 \end{bmatrix}$$

form the 'bosonic sector'; for example

$$\begin{bmatrix} \psi(x) \\ 0 \end{bmatrix}$$

is the ground state of H_s if ψ is the ground state of H. The states

$$\begin{bmatrix} 0 \\ g(x) \end{bmatrix}$$

form the 'fermionic sector', in which $\tilde{H}$ does not have the eigenvalue ε. Note that ψ and $\mathrm{d}^2\psi/\mathrm{d}x^2$ have opposite sign in the region where U is significant, giving

$$\frac{1}{\psi}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \left(\frac{1}{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 = \frac{\mathrm{d}U}{\mathrm{d}x} < 0 \qquad V = \tfrac{1}{2}U^2 + \frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}x} < \tfrac{1}{2}U^2 - \frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}x} = \tilde{V}. \tag{2.8}$$

The definitions of H and $\tilde{H}$ can always be adjusted so that $\varepsilon = 0$.

3. The supersymmetric harmonic oscillator

Take $U(x) = -x$, and ψ the ground state of $H = -\frac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + \frac{1}{2}x^2$ so that $\varepsilon = \frac{1}{2}$. Then H and $\tilde{H} = H + 1$ have the same eigenfunctions.

Van der Jeugt (1984) has associated the metaplectic representation of osp(3, 2) with a one-dimensional harmonic oscillator with spin. This work can be adapted to the supersymmetric oscillator by taking Q and $Q^\dagger$ as two of the odd operators, and using the osp(3, 2) commutators (see table 1) to construct the rest of the basis.

Take

$$R_{1-} = Q = \begin{bmatrix} 0 & 0 \\ A^- & 0 \end{bmatrix} \qquad R_{-1+} = Q^\dagger = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} \qquad A^\pm = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} - x\right). \tag{3.1}$$

Table 1. Commutators and anticommutators of a basis of osp(3, 2) (from Van der Jeugt (1984), omitting his subscripts $\frac{1}{2}$ so that $R_{0\pm\frac{1}{2}}$, for example, is $R_{0\pm}$).

	s_0	s_+	s_-	t_0	t_+	t_-	R_{1+}	R_{1-}	R_{-1+}	R_{-1-}	R_{0-}	R_{0-}
s_0	0	s_+	$-s_-$	0	0	0	R_{1+}	R_{1-}	$-R_{-1+}$	$-R_{-1-}$	0	0
s_+		0	$2s_0$	0	0	0	0	0	$\sqrt{2}\,R_{0+}$	$\sqrt{2}\,R_{0-}$	$\sqrt{2}\,R_{1+}$	$\sqrt{2}\,R_{1-}$
s_-			0	0	0	0	$\sqrt{2}\,R_{0+}$	$\sqrt{2}\,R_{0-}$	0	0	$\sqrt{2}\,R_{-1+}$	$\sqrt{2}\,R_{-1-}$
t_0				0	t_+	$-t_-$	$\frac{1}{2}R_{1+}$	$-\frac{1}{2}R_{1-}$	$\frac{1}{2}R_{-1+}$	$-\frac{1}{2}R_{-1-}$	$\frac{1}{2}R_{0+}$	$-\frac{1}{2}R_{0-}$
t_+					0	$2t_0$	0	R_{1+}	0	R_{-1+}	0	R_{0+}
t_-						0	R_{1-}	0	R_{-1-}	0	R_{0-}	0
R_{1+}							0	0	$-2t_+$	$-s_0+2t_0$	0	$2^{-1/2}s_+$
R_{1-}								0	s_0+2t_0	$2t_-$	$2^{-1/2}s_+$	0
R_{-1+}									0	0	0	$-2^{-1/2}s_-$
R_{-1-}										0	$-2^{-1/2}s_-$	0
R_{0+}											$2t_+$	$-2t_0$
R_{0-}												$-2t_-$

As above, $R_{\pm 1\mp}$ and

$$\begin{bmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{bmatrix} = s_0 + 2t_0$$

give a representation of S(2). Following van der Jeugt (1984) we take

$$s_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad s_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad s_0 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{3.2}$$

giving (using table 1)

$$R_{0\pm} = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} -A^{\pm} & 0 \\ 0 & A^{\pm} \end{bmatrix} \qquad R_{1+} = \begin{bmatrix} 0 & 0 \\ -A^+ & 0 \end{bmatrix} \qquad R_{-1-} = \begin{bmatrix} 0 & -A^- \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

$$t_+ = \frac{1}{2}\begin{bmatrix} A^+ & 0 \\ 0 & A^+ \end{bmatrix}^2 \qquad t_- = -\frac{1}{2}\begin{bmatrix} A^- & 0 \\ 0 & A^- \end{bmatrix}^2. \tag{3.4}$$

Since $t_+^\dagger = -t_-$, the sp(2) subalgebra corresponds to su(1, 1) rather than su(2). The general element of the representation space, graded by parity, is

$$\begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$$

where $\phi(x)$ belongs to the bosonic sector ($s_0 = -\frac{1}{2}$) and $\psi(x)$ belongs to the fermionic sector ($s_0 = \frac{1}{2}$). The action of the osp(3, 2) basis on eigenstates of H_s is illustrated in figure 1.

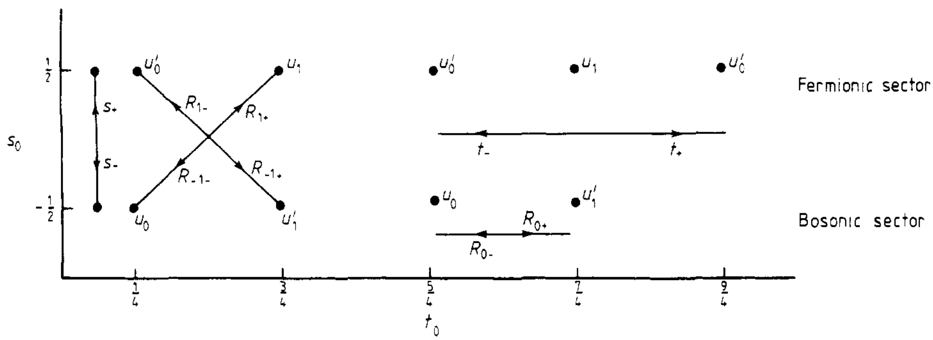

Figure 1. Action of the osp(3, 2) operators on supersymmetric oscillator eigenfunctions. In the bosonic sector ($s_0 = -\frac{1}{2}$) the energy is $2t_0$; in the fermionic sector ($s_0 = \frac{1}{2}$) the energy is $2t_0 + 1$. The symbols u and u' distinguish states of the different irreducible representation spaces $\mathcal{S}$ and $\mathcal{S}'$ of the osp(2, 2) subalgebra; the subscripts 0 and 1 indicate functions of even and odd grade (parity).

There is an osp(1, 2) subalgebra, spanned by t_0, $t_\pm$, $R_{0\pm}$, for which the bosonic sector and the fermionic sector are each irreducible representation spaces. This is the spectrum generating algebra for the ordinary harmonic oscillator with no supersymmetry.

There is an osp(2, 2) subalgebra spanned by s_0, t_0, $t_\pm$, $R_{1\pm}$, $R_{-1\pm}$; the osp(3, 2) representation branches to two irreducible representations of this subalgebra, each representation space containing the even functions of one sector and the odd functions of the other sector (cf figure 1). This osp(2, 2) subalgebra was given by de Crombrugghe and Rittenberg (1983) as the dynamical symmetry algebra. Note however that it does not contain the osp(1, 2) algebra associated with the ordinary harmonic oscillator. It does contain the osp(1, 2) algebra (Baake and Reinicke 1986) spanned by the even operators t_0, $t_\pm$ and the odd operators $(R_{1\pm} - R_{-1\pm})/2\sqrt{2}$, which Baake and Reinicke denoted by $V_\pm$. The representations $\mathscr{S}$ and $\mathscr{S}'$ remain irreducible on restriction to this osp(1, 2) subalgebra; the action of the operators on $\mathscr{S}$ is illustrated in figure 2. This branching rule is consistent with the corresponding result for finite-dimensional representations. The annihilation of the lowest state in $\mathscr{S}$ by $R_{-1\pm}$ indicates the representation corresponds to class (b) of Baake and Reinicke (1986), and the representation on $\mathscr{S}'$ corresponds to their class (c) as its lowest state is annihilated by $R_{1\pm}$. Table 2 shows the correspondence between bases for osp(2, 2).

Another discussion of superalgebras and the one-dimensional supersymmetric harmonic oscillator has been given by Beckers *et al* (1987). Their spectrum generating superalgebra contains the same osp(2, 2) subalgebra. The remaining four operators cannot be compared with $s_\pm$ and $R_{0\pm}$ because their representation space is not graded by parity, but by making odd operators convert bosonic sector states into fermionic sector states and vice versa.

There are also two five-dimensional subalgebras spanned by the even operators t_0, $t_\pm$ and the odd operators $R_{\alpha\pm}$ with either $\alpha = 1$ or $\alpha = -1$. These algebras are not semisimple since the odd operators form a solvable ideal. Their representations on $\mathscr{S}$ or $\mathscr{S}'$ are reducible but indecomposable, the invariant subspaces consisting of the fermionic states for $\alpha = 1$ and the bosonic states for $\alpha = -1$.

There are four S(2) subalgebras that illustrate various types of representation.

(i) Basis $H_s = s_0 + 2t_0$, $Q_\pm = R_{\pm 1\mp}$ (as in (2.4) and (2.5)): the ground state of the bosonic sector is a one-dimensional representation space in which all operators are zero; bosonic and fermionic states with the same energy E give a two-dimensional representation space in which the even operator H_s has the constant value $E - \frac{1}{2}$. The

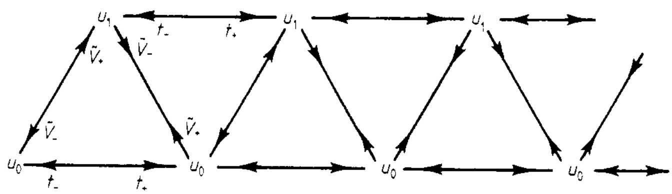

Figure 2. Action in the representation space $\mathscr{S}$ of the osp(1, 2) subalgebra in which the odd operators are $\tilde{V}_\pm = (R_{1\pm} - R_{-1\pm})/2\sqrt{2}$.

Table 2. Correspondence between bases for osp(2, 2) representations. The first two columns are the commuting even operators of the Cartan subalgebra. The first row is the standard basis of Scheunert *et al* (1977), designed to exhibit the isomorphic spl(1, 2). The last row is the basis used by Beckers *et al* (1987).

Standard basis	B	Q_3	Q_+	Q_-	V_+	V_-	W_+	W_-
$\mathscr{S}$ or $\mathscr{S}'$	$\frac{1}{2}s_0$	t_0	t_+	t_-	$2^{-1/2}R_{1+}$	$2^{-1/2}R_{1-}$	$-2^{-1/2}R_{-1+}$	$-2^{-1/2}R_{-1-}$
$\mathscr{S}_L$	$s_0 + t_0^- - t_0^+$	$t_0^- + t_0^+$	$2t_+^{-+}$	$2t_-^{-+}$	R_{1+}^+	R_{1-}^-	$-R_{-1+}^-$	$-R_{-1-}^+$
$\mathscr{S}_{-L}$	$s_0 - t_0^- + t_0^+$	$t_0^- + t_0^+$	$2t_+^{-+}$	$2t_-^{-+}$	R_{1+}^-	R_{1-}^+	$-R_{-1+}^+$	$-R_{-1-}^-$
BDH (1987) ($t = 0$)	$\frac{1}{2}H_F$	$\frac{1}{2}H_B$	$\frac{1}{2}\mathrm{i}C_+$	$\frac{1}{2}\mathrm{i}C_-$	$-2^{-1/2}\mathrm{i}S_+$	$-2^{-1/2}\mathrm{i}Q_+$	$-2^{-1/2}\mathrm{i}Q_-$	$-2^{-1/2}\mathrm{i}S_-$

two states are

$$f_1=\begin{bmatrix}\psi_k\\0\end{bmatrix}\qquad f_2=\begin{bmatrix}0\\\psi_{k-1}\end{bmatrix}\tag{3.5}$$

giving

$$H_s f_i=kf_i\qquad Q_-f_1=Q_+f_2=0\qquad Q_+f_1=\sqrt{k}f_2\qquad Q_-f_2=\sqrt{k}f_1\tag{3.6}$$

provided that ψ_k is a normalised oscillator eigenfunction so that

$$A_+\psi_k=(k+1)^{1/2}\psi_{k+1}\qquad A_-\psi_k=\sqrt{k}\,\psi_{k-1}.\tag{3.7}$$

(ii) Basis $-s_0+2t_0$, $R_{\pm1\pm}$: the ground state of the fermionic sector is a one-dimensional representation space in which all operators are zero; any state in the bosonic sector with energy $E-1$ is paired with the fermionic sector state with energy $E+1$ to give a two-dimensional representation space in which the even operator $-s_0+2t_0$ has the constant value $E-\frac{1}{2}$. These representations are equivalent to those in (i).

(iii) For the basis t_+, $R_{\pm1+}$ the representations are reducible, indecomposable and infinite dimensional; a representation space contains one of the representation spaces in (ii) and all states of higher energy that can be obtained using t_+. Alternatively a representation space is either $\mathscr{S}$ or $\mathscr{S}'$ (see figure 1) omitting all states below some minimum value of t_0. If this space has basis ($j=0,1,2,\ldots$)

$$v_j=\begin{bmatrix}\psi_{n+j}\\0\end{bmatrix}\quad(j\text{ even})\qquad v_j=\begin{bmatrix}0\\\psi_{n+j}\end{bmatrix}\quad(j\text{ odd})\tag{3.8}$$

then ($\alpha=\pm1$)

$$(t_+)v_j=\tfrac{1}{2}[(n+j+1)(n+j+2)]^{1/2}v_{j+2}\qquad(R_{\alpha+})v_j=-\tfrac{1}{2}\{\alpha+(-1)^j\}[(n+j+1)]^{1/2}v_{j+1}.\tag{3.9}$$

This is not a star representation. The alternative equivalent representation interchanges the sectors in (3.8).

(iv) For the basis t_-, $R_{\pm1-}$ an n-dimensional representation space is obtained from $\mathscr{S}$ or $\mathscr{S}'$ (see figure 1) by omitting all states with $t_0>\frac{1}{2}n-\frac{1}{4}$. Using $\mathscr{S}$ this basis is

$$u_1=\begin{bmatrix}\psi_0\\0\end{bmatrix}\qquad u_2=\begin{bmatrix}0\\\psi_1\end{bmatrix}\qquad u_3=\begin{bmatrix}\psi_2\\0\end{bmatrix}\qquad\ldots\qquad u_n=\begin{bmatrix}\psi_{n-1}\\0\end{bmatrix}\text{ or }\begin{bmatrix}0\\\psi_{n-1}\end{bmatrix}\tag{3.10}$$

and ($\alpha=\pm1$)

$$(t_-)u_{j+1}=-\tfrac{1}{2}(j^2-j)^{1/2}u_{j-1}\qquad(R_{\alpha-})u_{j+1}=\tfrac{1}{2}[\alpha+(-1)^j]j^{1/2}u_j.\tag{3.11}$$

The representation is irreducible if $n=1$ and otherwise reducible but indecomposable. It is not a star representation. The case $n=4$ is illustrated in figure 3(b), with the operator notation $H=t_-$, $Q_\pm=R_{\pm1-}$.

The representations (i) and (ii) are just the typical representations discussed by de Crombrugghe and Rittenberg (1983). The operator s_0 can be used for the automorphism M appearing in their discussion:

$$s_0f_1=-\tfrac{1}{2}f_1\qquad s_0f_2=\tfrac{1}{2}f_2\qquad s_0u_j=(-1)^j\tfrac{1}{2}u_j.\tag{3.12}$$

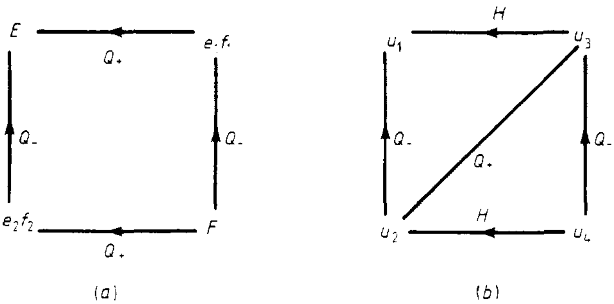

Figure 3. Different examples of four-dimensional reducible but indecomposable representations of S(2), showing the action of the even operator H and the odd operators $Q_{\pm}$; all operations not shown on the diagrams result in zero.

The tensor product of two typical two-dimensional representations of S(2) was given by de Crombrugghe and Rittenberg (1983). Their example of a reducible but indecomposable product representation is illustrated in figure 3(a), the functions E and F being certain linear combinations of e_1f_2 and e_2f_1. This is evidently inequivalent to the representation illustrated in figure 3(b).

4. The supersymmetric D-dimensional oscillator

For each of the operators in equations (3.1), (3.3) and (3.4) there are now D operators, one in each coordinate x_i:

$$A_i^{\mp}=2^{-1/2}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x_i}-x_i\right) \qquad R^i_{1-}=\begin{bmatrix}0 & 0\\ A_i^- & 0\end{bmatrix} \qquad t^i_{\pm}=\pm\frac{1}{2}\begin{bmatrix}A_i^{\mp} & 0\\ 0 & A_i^{\pm}\end{bmatrix}^2 \qquad \text{etc.} \tag{4.1}$$

The three operators in equation (3.2) are retained. Commutators involving different coordinates are zero, so that table 1 still supplies most of the results. However anticommutators produce $2D(D-1)$ operators t_0^{ij}, t_0^{ji}, $t_{\pm}^{ij}$, with diagonal elements $-A_i^+A_j^-$, $-A_i^-A_j^+$, $A_i^{\mp}A_j^{\mp}$ respectively. The $6D$ odd operators and $2D^2+D+3$ even operators form the superalgebra osp$(3,2D)$, which seems sufficient to generate all states of a supersymmetric D-dimensional oscillator with two components (bosonic and fermionic), although de Crombrugghe and Rittenberg (1983) suggest osp$(2D,2D)$ as the appropriate superalgebra. Again parity grades the representation space.

The discussion of subalgebras and their representations given for the one-dimensional oscillator will now be extended to the two-dimensional case. The action of the even operators is illustrated in figure 4; the ten operators $t^i_{\pm}$, t^i_0, t^{ij}_0 and $t^{12}_{\pm}$ form a basis for the algebra sp(4). The action of the odd operators is illustrated in figure 5; including with the sp(4) basis the four odd operators $R^i_{0\pm}$ gives the osp(1, 4) spectrum generating algebra for the ordinary (non-supersymmetric) oscillator (de Crombrugghe and Rittenberg 1983).

On restriction to the osp(2, 4) subalgebra with basis s_0, $R^i_{\alpha\pm}$ ($i=1,2$; $\alpha=\pm1$) and the sp(4) basis, the representation splits into two irreducible representation spaces containing the even (parity) states of one sector and the odd states of the other. This is essentially the same as the one-dimensional case.

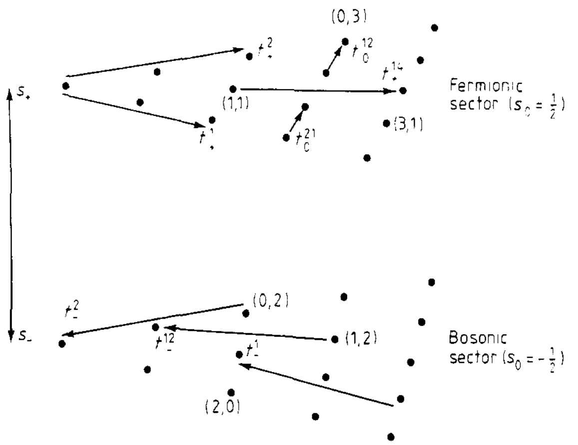

Figure 4. Action of the even osp(3, 4) operators on supersymmetric two-dimensional oscillator eigenfunctions. The sp(4) operators t_+^1, etc, have the same action in each sector. Numbers (n_1, n_2) show the number of applications of the creation operator in each variable; the eigenvalues of the Cartan subalgebra basis (t_0^1, t_0^2) are given by $(\frac{1}{2}n_1 + \frac{1}{4}, \frac{1}{2}n_2 + \frac{1}{4})$. The energy is $(n_1 + n_2 + 1)$ in the bosonic sector, or $(n_1 + n_2 + 2)$ in the fermionic sector.

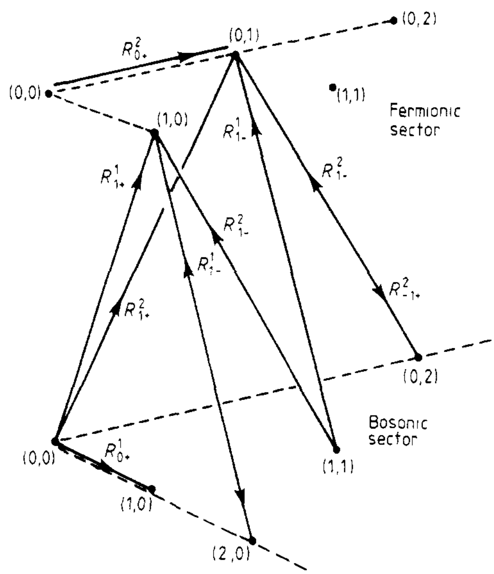

Figure 5. Action of the odd osp(3, 4) operators on supersymmetric two-dimensional oscillator eigenfunctions. In all cases changing the signs of the subscripts gives the operator with opposite action. The states are labelled with numbers (n_1, n_2) showing the number of oscillator quanta in each variable.

There are also representations of S(2) of the types (i)–(iv) considered in one dimension, and the extra dimension can be used to extend these. For example the seven-dimensional superalgebra with basis t_-^i, t_-^{12}, $R_{\alpha-}^i$ ($i = 1, 2$; $\alpha = \pm 1$) has reducible, indecomposable representations of type (iii) of dimension $\frac{1}{2}n(n+1)$. The states of the six-dimensional representation are shown connected by odd operators in figure 5. This superalgebra is not semisimple.

Supersymmetry for the D-dimensional oscillator has been discussed by Kostelecky *et al* (1985), using the one-dimensional formalism of § 2 with the radial coordinate only. To show how their work is related to the superalgebras defined above, consider the two-dimensional oscillator. It is necessary to replace the operators $A_i^\mp$ by the combinations $A_\pm = 2^{-1/2}(A_1^- \mp iA_2^-)$, $A_\pm^* = 2^{-1/2}(A_1^+ \pm iA_2^+)$ which effect unit changes in the angular momentum and the energy. The angular momentum is raised by A_- and by A_+^*. Then equations analogous to (4.1) define a new basis for the osp(3, 4) superalgebra, for example

$$R_{1-}^- = \begin{bmatrix} 0 & 0 \\ A_- & 0 \end{bmatrix} \qquad R_{-1+}^- = \begin{bmatrix} 0 & A_-^* \\ 0 & 0 \end{bmatrix} \qquad t_+^{-+} = \begin{bmatrix} A_-^* A_+^* & 0 \\ 0 & A_-^* A_+^* \end{bmatrix} \qquad \text{etc.} \tag{4.2}$$

The action of these operators is illustrated by relabelling figures 4 and 5: superscripts 1 and 2 are replaced by $-$ and $+$, and the state labels (N, m) show energy and angular momentum, as in figure 6. In polar coordinates

$$A_\pm = \tfrac{1}{2} e^{\mp i\phi}\left(r + \frac{\partial}{\partial r} \mp \frac{i}{r}\frac{\partial}{\partial \phi}\right) \qquad A_\pm^* = \tfrac{1}{2} e^{\mp i\phi}\left(r - \frac{\partial}{\partial r} \mp \frac{1}{r}\frac{\partial}{\partial \phi}\right). \tag{4.3}$$

The grading of the representation space is still determined by the physical parity, which is the parity of m; this is the same as the parity of N, and could be determined by the (unphysical) parity in r of the radial functions.

To identify operators equivalent to the Q and Q^+ used by Kostelecky *et al*, it is necessary to consider separately different signs of m, since the Q and Q^+ relate to the radial equation only, and involve $|m|$ rather than m. The shifts between sectors and the changes in angular momentum and energy identify the correspondences

$$\begin{aligned} &Q \leftrightarrow R_{1-}^- \qquad Q^+ \leftrightarrow R_{-1+}^- \qquad (m \geq 0) \\ &Q \leftrightarrow R_{1-}^+ \qquad Q^+ \leftrightarrow R_{-1+}^+ \qquad (m \leq 0). \end{aligned} \tag{4.4}$$

The pairs of states connected by this supersymmetry are shown in figure 6. Pairs having the same angular momentum in a sector lie in a plane in figure 6, and taking two more odd operators allows all states in such a plane to be obtained. Taking the anticommutators of the odd operators gives even operators, and their commutators close to form an osp(2, 2) superalgebra. For each integer m there is a representation space $\mathcal{S}_m$: for $m \geq 0$ this space is spanned by the states in figure 6(a) labelled $(m+2k, m)$

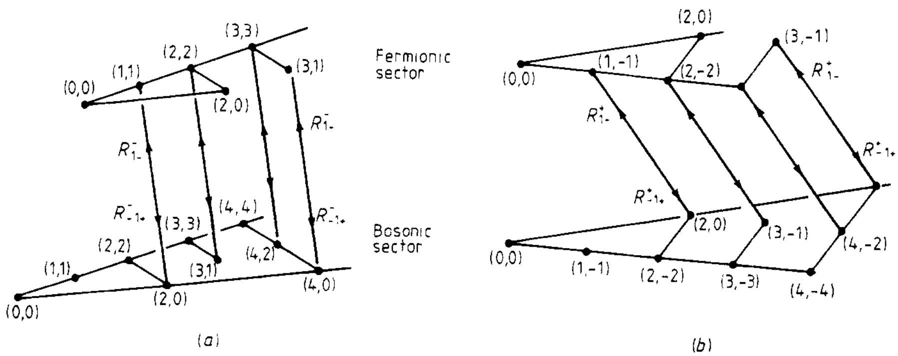

Figure 6. Action of odd operators which correspond to the charge operators used by Kostelecky *et al* (1985) for (a) $m \geq 0$, (b) $m \leq 0$. The state label is (N, m) where m is the angular momentum and the energy is $N+1$ in the bosonic sector and $N+2$ in the fermionic sector. States $(|m|, m)$ are annihilated.

in the bosonic sector and $(m-1+2k, m+1)$ in the fermionic sector; for $m<0$ the space is spanned by the states in figure 6(b) labelled $(|m|+2k, m)$ in the bosonic sector and $(|m|-1+2k, m-1)$ in the fermionic sector. The even and odd operators giving a standard basis of $\mathrm{osp}(2,2)\sim \mathrm{spl}(1,2)$ are shown in table 2.

As these representation spaces have constant angular momentum in each sector, it is sufficient to consider radial operators such as Q and $Q^{\dagger}$. These act on $U(r)$ where $U(r)\sqrt{r}$ is the actual radial function, so $\partial/\partial r$ in (4.3) should be replaced by $(\partial/\partial r)-(1/2r)$. Also $\partial/\partial\phi$ is replaced by $\mathrm{i}m$ when acting in the bosonic sector; in the fermionic sector $\partial/\partial\phi$ is replaced by $\mathrm{i}(m+1)$ if $m\geqslant 0$ and by $\mathrm{i}(m-1)$ if $m<0$. The resulting operators for the $\mathrm{osp}(2,2)$ basis are independent of the sign of m when expressed in terms of $L=|m|$.

Finally the dependence of these radial operators on the dimension D is always as in Q and $Q^{\dagger}$. Taking the $D=2$ results with $m>0$ and replacing m by $L+\frac{1}{2}D-1$ therefore gives an $\mathrm{osp}(2,2)$ algebra of operators acting on radial functions of the D-dimensional oscillator. The standard basis is

$$B=(-D-2L+1)I/4+s_0/2 \qquad Q_3=HI/2+(2L+D-1)(I+s_0)/8r^2 \tag{4.5}$$

$$Q_{\pm}=\tfrac{1}{2}(\mp H\pm r^2-r\partial/\partial r-\tfrac{1}{2})I+(2s_0\mp I)(2L+D-1)/8r^2 \tag{4.6}$$

$$U_{\pm}=[(\mp r+\partial/\partial r)/2-(2L+D-1)/4r]s_+$$

$$W_{\pm}=[(\mp r+\partial/\partial r)/2+(2L+D-1)/4r]s_- \tag{4.7}$$

where the matrices $s_{\pm}$ and s_0 are given in equation (3.2), I is the 2×2 unit matrix, and

$$H=(-\partial^2/\partial r^2+r^2)/2+(2L+D-1)(2L+D-3)/4r^2.$$

The grading of the representation space is now determined by (unphysical) parity in r. The $\mathrm{osp}(2,2)$ representations remain irreducible on restriction to $\mathrm{osp}(1,2)$ by omitting the even operator B and replacing the four odd operators by

$$V_{\pm 1/2}=\tfrac{1}{2}(V_{\pm}+W_{\pm})=\frac{1}{4}\begin{bmatrix}0 & \pm r+\mathrm{d}/\mathrm{d}r+(2L+D-1)/2r\\ \mp r+\mathrm{d}/\mathrm{d}r-(2L+D-1)/2r & 0\end{bmatrix}. \tag{4.8}$$

Only the $\mathrm{osp}(1,2)$ subalgebra is needed to generate the representation space (all radial functions with fixed angular momentum). However the supersymmetric Hamiltonian

$$H_{ss}=2B+2Q_3$$
$$=\frac{1}{2}\begin{bmatrix}-2L-D-\partial^2/\partial r^2+r^2+(2L+D-3)(2L+D-1)/4r^2 & 0\\ 0 & -2L-D+2-\partial^2/\partial r^2+r^2+(2L+D+1)(2L+D-1)/4r^2\end{bmatrix} \tag{4.9}$$

is in the $\mathrm{osp}(2,2)$ algebra but not in the $\mathrm{osp}(1,2)$ subalgebra.

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